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AN ALGORITHM FOR SHAPE OPTIMIZATION IN ELLIPTIC SYSTEMS*

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The problem of determining the shape of the simply connected cross-section of an elastic homogeneous prismatic rod which has the maximum torsional rigidity is considered. The cross-section has to belong to a given set in two-dimensional Euclidean space and the usual isoperimetric constraint is imposed on its area. A method of successive approximations is given for finding the shape. At each approximation the level line of the solution of a boundary value problem in a domain which is chosen in a special way from the previous approximation is taken.

Suppose that, in Euclidean space R^2, we are given the bounded closed set D. Let O be the set of all simply connected open domains G of R^2 which belong to the set D. For each domain G of O, bounded by the closed Jordan curve Γ, $U(\Gamma; p)$ is the solution of the boundary value problem (Δ is the Laplace operator)

$$-\Delta U(\Gamma; p) = 1,\ p \in G;\ U(\Gamma; p) = 0,\ p \in \Gamma \tag{1}$$

Suppose we are given the functional $J(\Gamma) = \int_G U(\Gamma;\ p)\, dp$ and the quantity P: $\operatorname{mes}(D) > P > 0$, where $\operatorname{mes}(G)$ is the Lebesgue measure of G. We wish to find the element G° of O (and its boundary Γ°) such that

$$J(\Gamma^\circ) = \sup\{J(\Gamma) \mid G \in O,\ \operatorname{mes}(G) = P\} \tag{2}$$

We know /1, 2/ that a necessary condition for optimality of the contour Γ° is

$$\begin{aligned} &|\nabla U(\Gamma^\circ;\ p)| = \lambda^2, \quad p \in \Gamma^\circ \setminus \partial D \\ &|\nabla U(\Gamma^\circ;\ p)| \geqslant \lambda^2,\ p \in \Gamma^\circ \cap \partial D \end{aligned} \tag{3}$$

Here, λ is a constant.

Note that this problem is only of interest when the set D and the quantity P are such that D does not include a domain of circular shape of measure P. Otherwise, the solution of problem (2) is obvious, see /3/.

We introduce the notation

$$\begin{gathered} \bar{G} = G \cup \Gamma,\ c(\Gamma) = \max\{U(\Gamma;\ p) \mid p \in \bar{G}\} \\ I(\Gamma) = (0,\ c(\Gamma)),\ \Gamma_c = \{p \in G \mid U(\Gamma;\ p) = c\} \\ B(\Gamma) = \{\Gamma_c \mid c \in I(\Gamma)\} \end{gathered}$$

Theorem 1. Let G be a domain of O. Then, given any closed Jordan contour $\Gamma^* \subseteq G$ and any $c \in I(\Gamma)$, we have

$$J(\Gamma_c) - \int_{R_c} \varphi^2(p)\, dp \geqslant J(\Gamma^*) - \int_{R^*} \varphi^2(p)\, dp \tag{4}$$

$$R_c = \bar{G}_c \setminus G^*; \quad R^* = \bar{G}^* \setminus G_c; \quad \varphi(p) = |\nabla U(\Gamma; p)|, \quad p \in G$$

Here, G_c and G^* are the domains bounded by Γ_c and Γ^* respectively.

For the proof, we use the method given in /4/. Given any $\alpha \leqslant \beta$ ($\alpha, \beta \in I(\Gamma)$), we define the sets

$$S_{\beta-\alpha} = G_\alpha \setminus G_\beta, \quad S^*_{\beta-\alpha} = G^* \cap S_{\beta-\alpha}$$

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$$G_\alpha^* = G_\alpha \cap G^*, \quad l_\alpha = \Gamma^* \cap G_\alpha$$
$$\gamma_\alpha = \Gamma_\alpha \cap G^*, \quad \Gamma_\alpha^* = l_\alpha \cup \gamma_\alpha$$

We fix some $c \in I(\Gamma)$.

The proof is in two stages.

1°. We first show that (4) holds under the auxiliary assumptions with respect to the contours Γ_c and Γ^*: b) $G_c \subset G^*$, b) Γ^* is a continuously differentiable curve which has not more than a finite number of points of intersection with any contour Γ_α, $\alpha \in I(\Gamma)$.

Put $R_\alpha^* = G_\alpha^* \setminus G_c$, where $\alpha \in I(\Gamma)$ (Fig.1).

We introduce the auxiliary function

$$\omega(\alpha) = J(\Gamma_\alpha^*) - \int_{R_\alpha^*} \varphi^2(p)\,dp, \alpha \in (c^*, c)$$
$$c^* = \inf\{\alpha \in I(\Gamma) \mid G^* \subseteq G_\alpha\}$$

Clearly,

$$\omega(c^*) = J(\Gamma^*) - \int_{R^*} \varphi^2(p)\,dp, \quad \omega(c) = J(\Gamma_c)$$

It is therefore sufficient for the proof to show that $\omega(c) \geqslant \omega(c^*)$.

We will show that the function $\omega(\alpha)$ is monotonically increasing in the interval (c^*, c). For this, we show that, given any $\alpha \in (c^*, c)$ and any $\delta \in (0, c-\alpha)$, we have

$$\omega(\alpha+\delta) - \omega(\alpha) \geqslant o(\delta) \tag{5}$$

We consider the equation

$$\omega(\alpha+\delta) - \omega(\alpha) = A - B - C$$
$$A = \int_{S_\delta^*} \varphi^2(p)\,dp, \quad B = \int_{G_{\alpha+\delta}^*} V(\Gamma_{\alpha+\delta}^*; p)\,dp$$
$$C = \int_{S_\delta^*} U(\Gamma_\alpha^*; p)\,dp, \quad S_\delta^* = S_\alpha^* \setminus \overline{S_{\alpha+\delta}^*}$$
$$V(\Gamma_{\alpha+\delta}^*; p) = U(\Gamma_\alpha^*; p) - U(\Gamma_{\alpha+\delta}^*; p)$$

Using Green's formula, we make the following transformations (the integration with respect to l is made over the set $\gamma_{\alpha+\delta}$):

$$B = -\int_{G_{\alpha+\delta}^*} V(\Gamma_{\alpha+\delta}^*; p)\,\Delta U(\Gamma_\alpha^*; p)\,dp = \int U(\Gamma_\alpha^*; p)\,D_n V(\Gamma_{\alpha+\delta}^*; p)\,dl -$$
$$\int U(\Gamma_\alpha^*; p)\,D_n U(\Gamma_\alpha^*; p)\,dl = \int U(\Gamma_\alpha^*; p)\,D_n(U(\Gamma_\alpha^*; p) - U(\Gamma_{\alpha+\delta}^*; p))\,dl -$$
$$\int U(\Gamma_\alpha^*; p)\,D_n U(\Gamma_\alpha^*; p)\,dl = \int |\nabla U(\Gamma_{\alpha+\delta}^*; p)|\,U(\Gamma_\alpha^*; p)\,dl$$

Here, D_n is the directional derivative along the outward normal to the relevant contour. Since $U(\Gamma; p) \geqslant 0$ in G /5/, it follows from the maximum principle for harmonic functions that, for any $\alpha \in (c^*, c)$, $p \in G_\alpha^* \cup \gamma_\alpha$, we have $U(\Gamma_\alpha; p) \geqslant U(\Gamma_\alpha^*; p)$. Here, obviously,

$$U(\Gamma_\alpha;\ p) = U(\Gamma;\ p) - \alpha \tag{6}$$

Hence, given any $\alpha \in (c^*, c)$ and $p \in \gamma_\alpha$, we have $|\nabla U(\Gamma_\alpha;\ p)| \geqslant |\nabla U(\Gamma_\alpha^*;\ p)|$. Hence

$$\omega(\alpha+\delta) - \omega(\alpha) \geqslant A - D - E$$
$$D = \int_{S_\delta^*} U(\Gamma_\alpha; p)\,dp, \ E = \int_{\gamma_{\alpha+\delta}} |\nabla U(\Gamma; p)|\,U(\Gamma_\alpha; p)\,dp \tag{7}$$

By Green's formula and (1), (2), (6), we have (Fig.2)

$$D = A - E - F$$
$$F = \int_{l_{\alpha,\delta}} U(\Gamma_\alpha; p)\,D_n U(\Gamma; p)\,dl, \ l_{\alpha,\delta} = l_\alpha \setminus \bar{l}_{\alpha+\delta} \tag{8}$$

Substituting Eq.(8) into condition (7), we obtain

$$\omega(\alpha+\delta)-\omega(\alpha)\geqslant F\geqslant-\delta H,\quad H=\int\limits_{l_{\alpha,\delta}}|\nabla U(\Gamma;p)|\,dl$$

since, by Eq.(6) and the definition of Γ_α $(\alpha\in I(\Gamma))$, we have on $l_{\alpha,\delta}$ the inequality $0<U(\Gamma_\alpha;\ p)<\delta$.

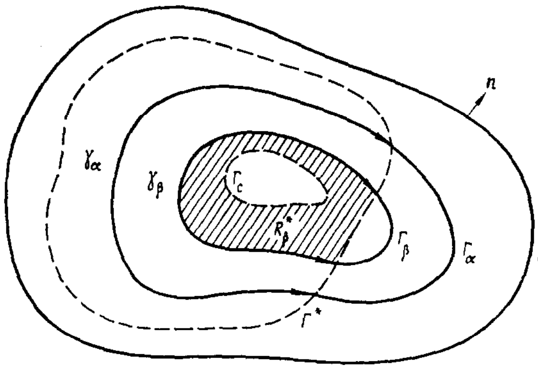

Fig.1

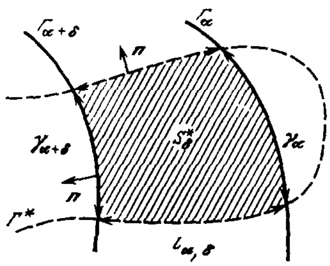

Fig.2

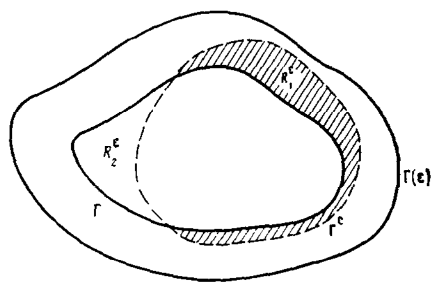

Fig.3

We calculate

$$0\leqslant\lim_{\delta\to0}\frac{1}{\delta}\{\delta H\}=\lim_{\delta\to0}H\leqslant M\lim_{\delta\to0}\operatorname{len}l_{\alpha,\delta}$$
$$0\leqslant M\leqslant\max\{|\nabla U(\Gamma;p)|\mid p\in\bar{G}_{c^*}\}$$

(len $l_{\alpha,\delta}$ is the length of $l_{\alpha,\delta}$). By condition b), we finally obtain $\lim\operatorname{len}l_{\alpha,\delta}=0$ as $\delta\to0$. Hence inequality (5) holds, which it was required to prove.

2°. Under the general assumptions made in the theorem, we can choose a sequence of domains $\{G_k{}^*\}_{k=1}^{\infty}$ such that, for any k, the contour $\Gamma_k{}^*$ which bounds the domain $G_k{}^*$ of the sequence satisfies condition b), while $\lim J(\Gamma_k{}^*)=J(\Gamma^*)$ as $k\to\infty$. We can thus exclude condition b) made at the first stage of the proof. On the other hand, we can show that, given any $\alpha\in I(\Gamma)$: $\alpha\leqslant c$, we have

$$J(\Gamma_\alpha)=J(\Gamma_c)+\int\limits_{S_{c-\alpha}}\varphi^2(p)\,dp$$

Hence follows the theorem for the contour Γ_α.

By the ε-neighbourhood of the domain G of O we mean the domain

$$G(\varepsilon)=\bigcup_{p\in\bar{G}}B(p,\varepsilon),\quad B(p,\varepsilon)=\{q\in R^2\,|\,|p-q|<\varepsilon\},\quad\varepsilon>0$$

Throughout, $\Gamma(\varepsilon)$ denotes the boundary of domain $G(\varepsilon)$. Consider a domain G of O. Let the contour Γ have the bounded curvature $0<\varkappa<\infty$, where $\varkappa=\varkappa(\Gamma)=\max\{\varkappa(\Gamma;p)\mid p\in\Gamma\}$; $\varkappa(\Gamma;p)$ is the curvature of Γ at the point p. Let $\varepsilon(\Gamma)=1/\varkappa(\Gamma)$.

In this case, we define for domain G in $(0, \varepsilon(\Gamma)]$ the function (Fig.3):

$$F(\Gamma;\varepsilon) = \frac{1}{\varepsilon^2}\left\{\int\limits_{R_1^\varepsilon} U^2(\varepsilon;p)\,dp - \int\limits_{R_2^\varepsilon} U^2(\varepsilon;q)\,dq\right\}$$

$$U(\varepsilon;p) = U(\Gamma(\varepsilon);p);\quad R_1^\varepsilon = G^\varepsilon \diagdown \bar{G},\quad R_2^\varepsilon = G \diagdown \overline{G^\varepsilon}$$

$$G^\varepsilon \in O : \Gamma^\varepsilon \in B(\Gamma(\varepsilon)),\quad \operatorname{mes} G^\varepsilon = \operatorname{mes} G$$

Clearly, $F(\Gamma;\ \varepsilon) > 0,\ \varepsilon \in (0,\ \varepsilon(\Gamma)]$.

Theorem 2. Let the domain G of O satisfy the following conditions: a) $\operatorname{mes} G = P$, b) the contour $\Gamma \in C^1$ and has the bounded curvature $\varkappa$, c) $F(\Gamma;\ \varepsilon) \geqslant C\varepsilon,\ C > 0$, d) there exists $\delta(\Gamma) \in (0,\ \varepsilon(\Gamma)]$ such that, for any $0 \leqslant \varepsilon \leqslant \delta(\Gamma)$, the curve Γ^ε is a simply connected contour. Then, there exists $0 < \varepsilon_0 \leqslant \delta(\Gamma)$ such that, given any $0 < \varepsilon \leqslant \varepsilon_0$, we have

$$J(\Gamma^\varepsilon) > J(\Gamma)$$

Before turning to the proof, we will make some preliminary remarks: first, for any domain G of O and many $Q < \operatorname{mes} G$, there exists $c \in I(\Gamma)$: $\operatorname{mes}(G_c) = Q$ /6/; and second, by the definition of $\varepsilon(\Gamma)$, $G(\varepsilon) \in O,\ \forall\varepsilon \in (0,\ \delta(\Gamma)]$.

Consider any $\varepsilon \in (0,\ \delta(\Gamma)]$. By the above, there exists $c(\varepsilon) \in I(\Gamma(\varepsilon))$ such that $\operatorname{mes} G_{c(\varepsilon)} = P$, and by Theorem 1, we have

$$J(\Gamma^\varepsilon) - \Phi_1 \geqslant J(\Gamma) - \Phi_2$$

$$\Phi_1 = \int\limits_{R_1^\varepsilon} \varphi^2(\varepsilon;p)\,dp,\quad \Phi_2 = \int\limits_{R_2^\varepsilon} \varphi^2(\varepsilon;p)\,dp$$

$$\varphi(\varepsilon;\ p) = |\nabla U(\varepsilon;\ p)|,\quad p \in G(\varepsilon)$$

We define the auxiliary function $\Phi(\Gamma;\ \varepsilon) = \Phi_1 - \Phi_2$. Note some of its properties: 1) $\Phi(\Gamma;\ 0) = 0$ for any domain $G \in O$, 2) the function $\Phi(\Gamma;\ \varepsilon)$ is continuous with respect to ε in $(0,\ \delta(\Gamma))$, see /7/.

To prove the theorem, it suffices to show that, for sufficiently small ε, we have

$$\Phi(\Gamma;\ \varepsilon) - \Phi(\Gamma;\ 0) = \Phi(\Gamma;\ \varepsilon) > 0 \tag{9}$$

Consider an arbitrary point p^* of R_1^ε. By the definition of $\varepsilon(\Gamma)$ and $\Gamma(\varepsilon)$, for the point p^* we can give on the contours Γ and $\Gamma(\varepsilon)$ the points p_1 and p_2 respectively such that p^* lies on the segment $L(p_1,\ p_2)$ which joins p_1 and p_2. Also, the segment $L(p_1,\ p_2)$ is perpendicular to the curves Γ and $\Gamma(\varepsilon)$ at their respective points.

By the formula of finite increments, we have

$$U(\varepsilon;\ p^*) = (\nabla U(\varepsilon;\ p_0),\ p_1 - p_2)\ |p^* - p_2|,\ p_0 \in L(p_1,\ p_2)$$

Hence, since $\varphi(\varepsilon;\ p)$ is continuous in $G(\varepsilon)$, we have

$$U(\varepsilon;\ p^*) = (\varphi(\varepsilon;\ p^*) + \psi(\varepsilon;\ p^*))\ |p^* - p_2|$$

Here, the function $\psi(\varepsilon;\ p)$ is continuous with respect to p in $M(\varepsilon)$, where $M(\varepsilon) = \overline{(G(\varepsilon) \diagdown G)} \cup \overline{R_2^\varepsilon}$; and $\psi(\varepsilon_i;\ p_i) \to 0$ as $i \to \infty$ for any sequences $\{\varepsilon_i\}_{i=1}^\infty$, $\{p_i\}_{i=1}^\infty$, such that $p_i \in M(\varepsilon_i)$ and $\varepsilon_i \to 0$ as $i \to \infty$.

Further, by the definition of R_1^ε, we have

$$U^2(\varepsilon;\ p^*) < \varepsilon^2(\varphi^2(\varepsilon;\ p^*) + \sigma(\varepsilon;\ p^*))$$

$$\sigma(\varepsilon;\ p) = 2\varphi(\varepsilon;\ p)\psi(\varepsilon;\ p) + \psi^2(\varepsilon;\ p)$$

We can show in the same way that $U^2(\varepsilon;\ q^*) > \varepsilon^2(\varphi^2(\varepsilon;\ q^*) + \sigma(\varepsilon;\ q^*))$, where q^* is any point of R_2^ε.

In all, we have

$$\varepsilon^2 F(\Gamma;\ \varepsilon) < \varepsilon^2\Phi(\Gamma;\ \varepsilon) + \varepsilon^3 K(\Gamma;\ \varepsilon)\operatorname{len}\Gamma$$

$$K(\Gamma;\ \varepsilon) = \max\{\sigma(\varepsilon;\ p)\ |\ p \in M(\varepsilon)\} - \min\{\sigma(\varepsilon;\ q)\ |\ q \in M(\varepsilon)\}$$

Hence $\Phi(\Gamma;\ \varepsilon) > (C - K(\Gamma;\ \varepsilon)\operatorname{len}\Gamma)\,\varepsilon$. It is now obvious that $K(\Gamma;\ \varepsilon) \to 0$ as $\varepsilon \to 0$, whence (9) follows.

It may be observed that condition d) of the theorem holds at least for domains on whose boundary the modulus of the gradient of the solution of problem (1) is greater than zero. As regards condition c), this is the hypothesis that the condition holds for domains which do not satisfy condition (3).

On the basis of Theorems 1 and 2, we give a numerical algorithm for constructing maximizing sequence of the domains $\{G_k\}_{k=0}^\infty$ in problem (2).

1) The initial domain G_0 can be found from intuitive considerations. But it is best to take $\rho(D;\ G_0) > 0$. Here, $\rho(D; G) = \min\{d(q, \Gamma) \mid q \in \partial D\}$, where $d(q, \Gamma) = \min\{|q-p| \mid p \in \Gamma\}$.

2) As the next term of the sequence we take the domain $G_k = G^{\varepsilon_{k-1}}$, where $\Gamma_k = \Gamma^{\varepsilon_{k-1}} \in B(\Gamma_{k-1}(\varepsilon_{k-1}))$.

3) We check the condition for ending the calculations:

$$\operatorname{mes}(\delta G_k) < \delta P \quad \text{or}$$
$$G_k : \forall \varepsilon_k > 0 \ \bar{G}_{k+1} \cap (R^2 \diagdown D) \neq \varnothing \tag{10}$$

where $\delta G_k = (G_k \diagdown G_{k-1}) \cup (G_{k-1} \diagdown G_k)$; $\delta P > 0$ is a pre-assigned quantity.

4) If condition (10) is not satisfied, we take $0 < \varepsilon_k < \delta(\Gamma_k)$ such that a) $\Phi(\Gamma_k;\ \varepsilon_k) > 0$; b) $G_{k+1} \subset D$ and pass to step 2).

We assume in the algorithm that all the domains G_k satisfy the constraints of Theorem 2.

Note that this algorithm is related to the numerical method used in /8/ for minimizing heat flow.

Numerical simulation shows that, along our sequence of domains, an improvement in fact occurs for the performance functional J. We see a decrease of $\rho(D; G_k)$ to zero and a fall in the error of the modulus of the gradient of the solution of problem (1) at the boundary of each successive domain. If the constraint $G \subset D$ is not essential, we see convergence of the sequence $\{G_k\}_{k=0}^{\infty}$ to a circle. Our method can also be used for some related problems (optimization of a doubly-connected section /9/, or minimization of a heat flow /4, 10/).

The main attention has been paid in the literature to optimization methods in problems without constraints of the inclusion type $G \subset D$ on the shape of the rod cross-section. Some of these algorithms may be found in /1, 2, 9/.

It must be said that the methods in /1, 2, 9/ make explicit use of the necessary condition (3) for optimality; an improvement in the accuracy of approximation to the optimal contour involves either an increase in the dimensionality of the non-linear system of algebraic equations which is solved in /2, 9/, or the need in /1/ to consider more and more complicated boundary value problems; when there are constraints of the type $G \subset D$ on the shape of the rod cross-section, it is doubtful whether these methods can be used.

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